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LETTER TO THE EDITOR

Series estimate of the exponent β for bond percolation on a simple cubic lattice

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Abstract. The critical exponent β for the bond problem on a simple cubic lattice is analysed by a Padé approximant technique using high-density expansions corresponding to three different definitions of the percolation probability. It is found that $\beta = 0.463 \pm 0.013 - 12\Delta p_c$ where $\Delta p_c = p_c - 0.247$. The need for more precise estimates of the critical density p_c is emphasised.

There is some disagreement in estimates of the value of the exponent β characterising the critical behaviour of the percolation probability for three-dimensional lattices. The estimates which range from 0.39 to 0.47—a difference amounting to almost 20% of the average value—are summarised in table 1. Universality arguments (Essam 1980) and the covering lattice transformation (Essam 1972) suggest that the exponent should be lattice independent and the same for bond and site percolation. Renormalisation group techniques have not, as yet, yielded reliable estimates of β; the ε expansion is known to order ε^2 but, since it starts from six dimensions, can only give qualitative results (Stauffer 1979, Essam 1980).

Table 1. Summary of direct estimates of β for percolation processes on three-dimensional lattices.

Estimate of β	Lattice	Bond/site	Method	Reference
0.39 ± 0.02	SC	site	Monte Carlo	Kirkpatrick (1976)
0.41 ± 0.01	SC	site	Monte Carlo	Sur *et al* (1976)
0.42 ± 0.06	FCC	site	Series	Sykes *et al* (1976)
0.47 ± 0.02	FCC	bond	Series	Blease *et al* (1976)

For bond percolation the percolation probability can be defined in a number of ways:

(i) the probability, $P(p, 0)$, that a randomly chosen lattice site is a member of an infinite bond cluster at bond density p,

(ii) the probability, $P(p, 1)$, that a randomly chosen site, which is an end point of at least one occupied bond, is a member of an infinite bond cluster,

(iii) the probability, $\bar{P}(p)$, that a randomly chosen occupied bond is a member of an infinite bond cluster.

Blease *et al* (1976) have proved that definitions (i) and (iii) lead to the same critical density and exponent, while the corresponding results for definitions (i) and (ii) have been proved by Whittington *et al* (1981). Hence, all three functions are characterised by the same critical density p_c and exponent β.

We have derived high-density expansions in powers of $q = 1-p$ for each of these three functions for bond percolation on a simple cubic (SC) lattice:

$$\begin{aligned}P(p,0) = 1 - q^6 - 6q^{10} + 6q^{11} - 45q^{14} + 90q^{15} - 57q^{16} - 260q^{18} + 900q^{19} \\ - 1200q^{20} + 572q^{21} - 1098q^{22} + 6360q^{23} - 14332q^{24} + 15444q^{25} \\ - 12450q^{26} + 39366q^{27} - 124284q^{28} + 218028q^{29} - 256649q^{30} + \ldots\end{aligned} \tag{1}$$

$$\begin{aligned}P(p,1) = 1 - 6q^{10} + 6q^{11} - 45q^{14} + 90q^{15} - 63q^{16} + 6q^{17} - 260q^{18} + 900q^{19} - 1245q^{20} \\ + 662q^{21} - 1161q^{22} + 6366q^{23} - 14592q^{24} + 16344q^{25} \\ - 13695q^{26} + 40028q^{27} - 125445q^{28} + 224394q^{29} - 271241q^{30} + \ldots\end{aligned} \tag{2}$$

$$\begin{aligned}\bar{P}(p) = 1 - q^{10} - 10q^{14} + 10q^{15} - 4q^{16} - 71q^{18} + 158q^{19} - 163q^{20} + 64q^{21} - 402q^{22} \\ + 1496q^{23} - 2608q^{24} + 2408q^{25} - 3347q^{26} + 11616q^{27} \\ - 28170q^{28} + 41536q^{29} - 53111q^{30} + 109460q^{31} \\ - 265894q^{32} + 491376q^{33} - \ldots .\end{aligned} \tag{3}$$

Each of the three series alternates in sign indicating a radius of convergence associated with a non-physical singularity on the negative real axis. We have estimated p_c and β by forming Padé approximants to the Dlog series (Gaunt and Guttmann 1974). Approximants based on relatively few terms may be misleading because of the irregular behaviour of the initial coefficients. After discarding such approximants and all those exhibiting defects (Gaunt and Guttmann 1974), we display the remaining 'usable' estimates as pole-residue plots in figure 1. A large number of diagonal Padé sequences, $[n+j/n]$ with fixed j, were studied. Most of the 'usable' estimates fall, irrespective of the value of j, on one of three reasonably well defined curves corresponding to the three definitions of the percolation probability.

As is well known, considerable care must be exercised in interpreting plots such as those in the figure. For example, for the bond problem on the triangular lattice (Blease *et al* 1978), the pole-residue plots for $P(p,0)$ and $\bar{P}(p)$ intersect at a value of q which is 0.003 above the exact critical value. The corresponding plots for the face-centred cubic (FCC) lattice (Blease *et al* 1976) do not intersect at all but run roughly parallel, the region of closest approach lying about 0.0015 above the best numerical estimate of q_c as obtained using low-density series. Nevertheless, the intersection points of the three curves in the figure clearly indicate the general region of interest, and a reasonably good estimate of β could be obtained if p_c were known. Since p_c is not known, we have used the biased estimate

$$p_c(\text{SC, bond}) = 0.247 \pm 0.003 \tag{4}$$

obtained by Sykes *et al* (1976) from an analysis of low-density series for the mean size of finite clusters assuming $\gamma = 1.66 \pm 0.11$. The uncertainties in (4) are large enough to embrace the only other recent estimates of p_c of which we are aware, namely

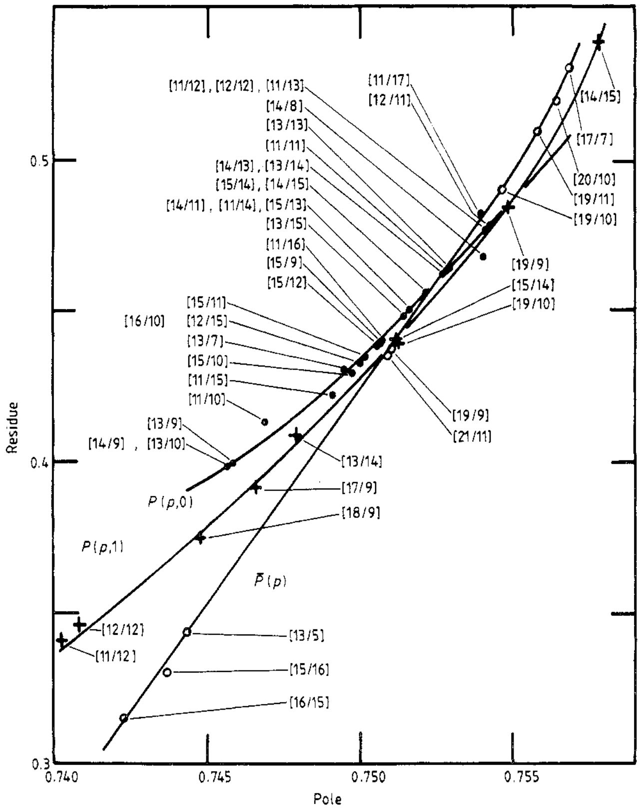

Figure 1. Bond percolation, simple cubic lattice. Pole-residue plots for percolation probabilities: ●, $P(p, 0)$; +, $P(p, 1)$; ○, $\bar{P}(p)$.

$p_c = 0.2465 \pm 0.0002$ (Fisch 1977) and $p_c = 0.249$ quoted in the review by Stauffer (1979). We find that a linear approximation in the vicinity of $p_c = 0.247$ gives

$$\beta(\text{sc, bond}) = 0.463 \pm 0.013 - 12\Delta p_c \tag{5}$$

where $\Delta p_c = p_c - 0.247$. (As an internal consistency check on (5) we have estimated β by evaluating Padé approximants to the series $(q - q_c)(\mathrm{d}/\mathrm{d}q) \ln P(\text{or } \bar{P})$ for a range of p_c values.) Assuming that $|\Delta p_c| \leq 0.003$ as in (4) we obtain from (5) the estimate

$$\beta(\text{sc, bond}) = 0.46 \pm 0.05. \tag{6}$$

Following Blease *et al* (1978) we have taken in (5) the separation of the plots over the interval defined by (4) as giving a realistic estimate of the inherent uncertainties (± 0.013) of the method. The fact that our final uncertainties in (6) are almost four times larger is simply a reflection of the large uncertainties in our estimate of p_c. (An uncertainty of 1% in p_c produces an uncertainty in β of almost $6\frac{1}{2}$%.)

Our estimate of β provides support for the estimate $\beta = 0.47 \pm 0.02$ of Blease *et al* (1976) which, until the present work, appeared anomalously high. It now seems to us that the uncertainties associated with the Monte Carlo estimates in table 1 are unrealistically small. Even so, there is a definite tendency for site problems to yield lower estimates of β (with central values clustered around 0.39 to 0.42) than do bond problems (where the corresponding range is now 0.46 to 0.47). There is also a secondary tendency in both the site and bond problems, for the simple cubic lattice to give lower estimates of β than the face-centred cubic lattice. These variations are probably caused by convergence difficulties associated with limited data rather than by a breakdown in universality. Unfortunately, we have no *a priori* way of knowing whether the site or bond problem nor which lattice is likely to provide the most reliable estimate. Assuming that the uncertainties in the series estimates in table 1 and in equation (6) are realistic, and that β is universal, we obtain as an overall estimate

$$\beta(\text{overall}) = 0.465 \pm 0.015. \tag{7}$$

In conclusion, we should like to emphasise the need for better estimates of p_c for both bond and site problems on three-dimensional lattices. It should be noted that the estimate $p_c = 0.2465 \pm 0.0002$ due to Fisch (1977) has uncertainties of only 0.08%. This estimate was obtained by analysing the first ten terms of the low-density expansion for the resistive susceptibility of a diluted resistor network (Stauffer 1979). The relatively large value of the appropriate critical exponent $\gamma^{(r)} \simeq 2.78$ (cf $\gamma \simeq 1.66$ for percolation) allows a more precise determination of p_c. If Fisch's estimate could be independently verified, then using (5) we could replace (6) by the revised estimate

$$\beta(\text{SC, bond}) = 0.469 \pm 0.015 \tag{8}$$

in excellent agreement with the estimate of $\beta = 0.47 \pm 0.02$ by Blease *et al* (1976) for the face-centred cubic bond problem. Perhaps the recent Monte Carlo method, based on finite size scaling, used by Vicsek and Kertész (1981) to estimate p_c with an estimated precision of 0.12% for the honeycomb site problem can be successfully adapted to three-dimensional lattices, especially to the simple cubic bond problem. Only when agreed, precise estimates of p_c are available will direct and precise estimates of critical exponents by series or Monte Carlo methods become a possibility.

Acknowledgments

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